MTH 520/622: Introduction to Hyperbolic Geometry
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1 Preliminaries

1.1 The upper half plane $\mathbb{H}$

(i) As a set, the *upper half plane* is given by

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

(ii) A *hyperbolic line* in the *upper half plane model* is defined to be one of the following two types of subsets of $\mathbb{H}$.

  (1) The intersection of a Euclidean line perpendicular to the real line $\mathbb{R}$ (i.e the X-axis) with $\mathbb{H}$.

  (2) The intersection of a Euclidean circle centered on the real line $\mathbb{R}$ (i.e the X-axis) with $\mathbb{H}$.

(iii) Two hyperbolic lines are said to be *parallel* if they do not intersect in $\mathbb{H}$.

(iv) Given a line $\ell \subset \mathbb{H}$ and point a $p \in \mathbb{H} \setminus \ell$, there exists infinitely many hyperbolic lines passing through p and parallel to ℓ. Consequently, the fifth postulate of Euclidean geometry does not hold true in hyperbolic geometry.

1.2 The Riemann sphere $\hat{\mathbb{C}}$

(i) As a set, the Riemann sphere $\hat{\mathbb{C}}$ is the union

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

(ii) There is a natural extension of the stereographic projection $p : S^2 \setminus \{N\} \to \mathbb{C}$ to a map $\bar{p} : S^2 \to \hat{\mathbb{C}}$ defined by

$$\bar{p}|_{\mathbb{C}} = p \text{ and } \bar{p}(N) = \infty,$$

which is a homeomorphism. Hence, topologically $\hat{\mathbb{C}} \approx S^2$ via $\bar{p}$.

(iii) For a point $z \in \hat{\mathbb{C}}$, an *open ball (or disk) $B_\epsilon(z)$ of radius ϵ centered at z* is defined by

$$B_\epsilon(z) = \begin{cases} \{w \in \mathbb{C} : |w - z| < \epsilon\}, & \text{if } z \in \mathbb{C}, \text{ and} \\ \{w \in \mathbb{C} : |w| > \epsilon\} \cup \{\infty\}, & \text{if } z = \infty. \end{cases}$$

(iv) We define

$$\text{Homeo}(\hat{\mathbb{C}}) = \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} : f \text{ is a homeomorphism}\}.$$

The set $\text{Homeo}(\hat{\mathbb{C}})$ forms a group under composition.

(v) Example: The function $r : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$r(z) = \begin{cases} 1/z, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = 0, \text{ and} \\ 0, & \text{if } z = \infty \end{cases}$$

is an element of $\text{Homeo}(\hat{\mathbb{C}})$.

(vi) A *circle* in $\hat{\mathbb{C}}$ either a Euclidean circle or the union of a Euclidean line with $\{\infty\}$.

(vii) Example: The set $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is a circle in $\hat{\mathbb{C}}$.

(viii) The set $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is called the *boundary at infinity* of hyperbolic space $\mathbb{H}$. Topologically, we can see that $\bar{\mathbb{R}} \approx S^1$ by naturally extending the stereographic projection in dimension 1.

(ix) For a set $A \subset \mathbb{H}$, the boundary at infinity is defined by

$$\partial_\infty(A) := \bar{A} \cap \bar{\mathbb{R}},$$

where $\bar{A}$ is the closure of A in $\hat{\mathbb{C}}$.

(x) Two hyperbolic lines ℓ_1 and ℓ_2 are said to be *ultraparallel* if

$$\partial_\infty(\ell_1) \cap \partial_\infty(\ell_2) = \emptyset.$$

(xi) Given a point $p \in \mathbb{H}$ and a point $q \in \bar{\mathbb{R}}$, there exists a unique hyperbolic line ℓ passing through p such that $\partial_\infty(\ell) = \{q\}$. Consequently, there is a unique circle in $\hat{\mathbb{C}}$ that contains a line ℓ in $\mathbb{H}$.

2 The general Möbius group

2.1 The group $\text{Möb}^+(\hat{\mathbb{C}})$

(i) We define

$$\text{Homeo}^c(\hat{\mathbb{C}}) = \{f \in \text{Homeo}(\hat{\mathbb{C}}) : \text{f maps circles in } \hat{\mathbb{C}} \text{ to circles in } \hat{\mathbb{C}}\}.$$

(ii) $\text{Homeo}^c(\hat{\mathbb{C}})$ forms a group under composition.

(iii) Examples:

(a) The map $r \in \text{Homeo}^c(\hat{\mathbb{C}})$.

(b) For $a, b \in \mathbb{C}$ and $a \neq 0$, consider the map $f \in \text{Homeo}(\hat{\mathbb{C}})$ defined by

$$f(z) = az + b, \text{ for } z \in \mathbb{C} \text{ and } f(\infty) = \infty.$$

Then $f \in \text{Homeo}^c(\hat{\mathbb{C}})$.

(iv) A Möbius transformation is a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$m(z) = \frac{az + b}{cz + d}, \text{ where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

(v) Consider a Möbius transformation $m(z) = \dfrac{az + b}{cz + d}$.

(a) If $c = 0$, then $m(z) = \dfrac{a}{d}z + \dfrac{b}{d}$.

(b) If $c \neq 0$, then $m(z) = (f \circ r \circ g)(z)$, where $g(z) = c^2 z + cd$ and $f(z) = -(ad - bc)z + \frac{a}{c}$, and $f(\infty) = \infty = g(\infty)$.

Consequently, $m \in \text{Homeo}^c(\hat{\mathbb{C}})$.

(vi) The set of all Möbius transformations on $\hat{\mathbb{C}}$ forms a group under composition, which we denote by $\text{Möb}^+(\hat{\mathbb{C}})$. Clearly, $\text{Möb}^+(\hat{\mathbb{C}}) \subset \text{Homeo}^c(\hat{\mathbb{C}})$.

(vii) If $m \in \text{Möb}^+(\hat{\mathbb{C}})$ fixes any three distinct points in $\hat{\mathbb{C}}$, then m is the identity.

(viii) Given four distinct points z_1, z_2, z_3, and z_4 in $\mathbb{C}$, we define the *cross ratio* of z_1, z_2, z_3, and z_4 by

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}.$$

If one of the z_k (say z_1) equals ∞, then we define the cross ratio by continuity, that is,

$$[\infty, z_2; z_3, z_4] = \lim_{z \to \infty} [z, z_2; z_3, z_4] = \frac{z_3 - z_2}{z_3 - z_4}.$$

(ix) Given a triple (z_1, z_2, z_3) of distinct points in $\hat{\mathbb{C}}$, there exists a unique $m \in \text{Möb}^+(\hat{\mathbb{C}})$ such that $(m(z_1), m(z_2), m(z_3)) = (0, 1, \infty)$, which is given by

$$m(z) = [z, z_3; z_2, z_1].$$

Consequently, the natural action of $\text{Möb}^+(\hat{\mathbb{C}})$ on the set of $\mathcal{T}$ of triple of distinct points in $\hat{\mathbb{C}}$ is uniquely transitive.

(x) $\text{Möb}^+(\hat{\mathbb{C}})$ acts transitively on the set $\mathcal{C}$ of circles in $\hat{\mathbb{C}}$, and on the set $\mathcal{D}$ of disks in $\hat{\mathbb{C}}$.

2.2 Classification of Möbius transformations

(i) Two Möbius transformations $m_1, m_2 \in \text{Möb}^+(\hat{\mathbb{C}})$ are said to be *conjugate* if there exists $p \in \text{Möb}^+(\hat{\mathbb{C}})$ such that $m_2 = p \circ m_1 \circ p^{-1}$.

(ii) As $\frac{az+b}{cz+d} = z$ yields a quadratic equation in z, an $m \in \text{Möb}^+(\hat{\mathbb{C}})$ can have at most 2 fixed points in $\hat{\mathbb{C}}$.

(iii) A Möbius transformation $m \in \text{Möb}^+(\hat{\mathbb{C}})$ is said to be:

(a) *parabolic*, if has only one fixed point in $\hat{\mathbb{C}}$ and is conjugate to the map $m'(z) = z + 1$.

(b) *elliptic*, if has two fixed points in $\hat{\mathbb{C}}$ and is conjugate to the map $m'(z) = az$, where $|a| = 1$, that is, $a = e^{i2\theta}$, for some $\theta \in [0, \pi)$.

(c) *loxodromic*, if has two fixed points in $\hat{\mathbb{C}}$ and is conjugate to the map $m'(z) = az$, where $|a| \neq 1$, that is, $a = re^{i2\theta}$, for some $r > 0$ and $\theta \in [0, \pi)$.

(iv) We can view a Möbius transformation $m(z) = \frac{az+b}{cz+d}$, as the map

$$z \overset{m}{\mapsto} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}, \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

Hence, there exists a natural surjective map

$$\varphi : \mathrm{GL}(2, \mathbb{C}) \to \text{Möb}^+(\hat{\mathbb{C}})$$

defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\varphi}{\mapsto} (z \mapsto \frac{az+b}{cz+d}),$$

where φ is a homomorphism. Moreover, we have that

$$\operatorname{Ker} \varphi = \{kI_2 : k \in \mathbb{C}\}.$$

Consequently,

$$\text{Möb}^+(\hat{\mathbb{C}}) \cong \mathrm{PGL}(2, \mathbb{C}) = \mathrm{PSL}(2, \mathbb{C}).$$

(v) Given a Möbius tranformation $m(z) = \frac{az+b}{cz+d}$, the equivalent Möbius tranformation

$$m(z) = \frac{\frac{a}{D}z + \frac{b}{D}}{\frac{c}{D}z + \frac{d}{D}}, \text{ where } D = ad - bc,$$

is called the *normalized form* of m.

(vi) Given a Möbius tranformation $m(z) = \frac{az+b}{cz+d}$ in its normalized form, we define

$$\mathrm{Trace}^2(m) := (a + d)^2.$$

(vii) Let m be a Möbius transformation that is not the identity. Then:

(i) m is parabolic if, and only if $\mathrm{Trace}^2(m) = 4$.

(ii) m is elliptic if, and only if $\mathrm{Trace}^2(m) \in [0, 4)$.

(iii) m is loxodromic if, and only if either $\mathrm{Im}(\mathrm{Trace}^2(m)) \neq 0$ or $\mathrm{Trace}^2(m) \in (-\infty, 0) \cup (4, \infty)$.

2.3 The group $\text{Möb}(\hat{\mathbb{C}})$

(i) The *complex conjugation* map $\mathscr{C} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined by

$$\mathscr{C}(z) = \bar{z}, \text{ for } z \in \mathbb{C} \text{ and } \mathscr{C}(\infty) = \infty.$$

(ii) The map $\mathscr{C}$ is a reflection through the circle $\bar{\mathbb{R}} \subset \hat{\mathbb{C}}$ and clearly, $\mathscr{C} \in \text{Homeo}^c(\hat{\mathbb{C}}) \setminus \text{Möb}^+(\hat{\mathbb{C}})$.

(iii) Given a circle $A \subset \hat{\mathbb{C}}$, consider a $m \in \text{Möb}(\hat{\mathbb{C}})$ such that $m(\bar{\mathbb{R}}) = A$. Then we define a *reflection through* A as the map

$$\mathscr{C}_A(z) = (m \circ \mathscr{C} \circ m^{-1})(z), \text{ for } z \in \hat{\mathbb{C}}.$$

(a) Note that $\mathscr{C}_A(z)$ is well defined as its independent of the choice of m.

(b) Every element in $\text{Möb}(\hat{\mathbb{C}})$ is a composition of finitely many reflections through circles.

(iv) The *general Möbius group* $\text{Möb}(\hat{\mathbb{C}})$ is defined as the subgroup of $\text{Homeo}^c(\hat{\mathbb{C}})$ generated by $\text{Möb}^+(\hat{\mathbb{C}})$ and $\mathscr{C}$. Thus, by definition, $\text{Möb}(\hat{\mathbb{C}}) \subset \text{Homeo}^c(\hat{\mathbb{C}})$.

(v) Every $m \in \text{Möb}(\hat{\mathbb{C}})$ either has the form $m(z) = \frac{az+b}{cz+d}$ or has the form $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

(vi) $\text{Möb}(\hat{\mathbb{C}}) = \text{Homeo}^c(\hat{\mathbb{C}})$.

(vii) Let S_1 and S_2 be surfaces. Then a map $f : S_1 \to S_2$ is said to be *conformal* if it preserves the angles, that is, given a two curves c_1 and c_2 in S_1 that intersect at $P \in S_1$ with an angle θ, $f(c_1)$ and $f(c_2)$ intersect at the same angle θ at $f(P)$.

(viii) A conformal map on an oriented surface is said to be *directly conformal* if it preserves orientation, and *indirectly conformal* if it reverses orientation.

(ix) Examples of conformal maps.

(a) Any self-homeomorphism of a closed orientable surface that is realizable as a rotation of the surface about an axis is directly conformal.

(b) Any self-homeomorphism of a closed orientable surface that is realizable as a reflection of the surface through a plane is indirectly conformal.

(c) The $\mathscr{C}$ is an indirectly conformal map of $\hat{\mathbb{C}}$.

(x) A map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is conformal at ∞ if, and only if $r \circ f$ is conformal at 0.

(xi) Every element $m \in \text{Möb}^+(\hat{\mathbb{C}})$ is directly conformal, while element $m \in \text{Möb}(\hat{\mathbb{C}}) \setminus \text{Möb}^+(\hat{\mathbb{C}})$ is indirectly conformal.

2.4 The groups Möb($\mathbb{H}$) and Möb$^+$($\mathbb{H}$)

(i) We define

(a) $\text{Möb}(\mathbb{H}) = \{m \in \text{Möb}(\hat{\mathbb{C}}) \,|\, m(\mathbb{H}) = \mathbb{H}\}$.

(b) $\text{Möb}^+(\mathbb{H}) = \{m \in \text{Möb}^+(\hat{\mathbb{C}}) \,|\, m(\mathbb{H}) = \mathbb{H}\}$.

(c) $\text{Möb}(\bar{\mathbb{R}}) = \{m \in \text{Möb}(\hat{\mathbb{C}}) \,|\, m(\bar{\mathbb{R}}) = \bar{\mathbb{R}}\}$.

(d) $\text{Möb}^+(\bar{\mathbb{R}}) = \{m \in \text{Möb}^+(\hat{\mathbb{C}}) \,|\, m(\bar{\mathbb{R}}) = \bar{\mathbb{R}}\}$.

(ii) Every element of $\text{Möb}(\bar{\mathbb{R}})$ has one of the following forms:

(a) $m(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ (or $i\,\mathbb{R}$) and $ad - bc = 1$.

(b) $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ with $a, b, c, d \in \mathbb{R}$ (or $i\,\mathbb{R}$) and $ad - bc = 1$.

(iii) Every element of $\text{Möb}^+(\mathbb{H})$ has the form

$$m(z) = \frac{az+b}{cz+d} \text{ with } a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1,$$

while every element of $\text{Möb}(\mathbb{H}) \setminus \text{Möb}^+(\mathbb{H})$ has the form

$$m(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \text{ with } a, b, c, d \in i\,\mathbb{R} \text{ and } ad - bc = 1.$$

Consequently, $\text{Möb}^+(\mathbb{H}) \cong \text{PSL}(2, \mathbb{R})$.

3 Hyperbolic geometry

3.1 The upper-half plane model $\mathbb{H}$

(i) We define the *metric for the upper half plane model* by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

(ii) If $\gamma : [a, b] \to \mathbb{H}$ is a path in $\mathbb{H}$ that is parametrized in $[a, b]$ with $\gamma(t) = x(t) + iy(t)$, then the *length* $\ell(\gamma)$ *of the path* γ is defined by

$$\ell_{\mathbb{H}}(\gamma) := \int_a^b \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

(iii) Given two points $P, Q \in \mathbb{H}$, the *distance* $d_{\mathbb{H}}(P, Q)$ between P and Q is defined by

$$d_{\mathbb{H}}(P, Q) := \inf \ell_{\mathbb{H}}(\gamma),$$

where the infimum is taken over all paths joining P and Q.

(iv) Let (M, d) be a metric space, and let $I = [a, b]$. A path $\gamma : I \to M$ is said to be a *geodesic* from a to b if there is a constant $c \geq 0$ such that for any $t \in I$ there exists a neighborhood J of t in I such that for any $t_1, t_2 \in J$, we have

$$d(\gamma(t_1), \gamma(t_2)) = c\, |t_1 - t_2|\, .$$

In other words, the path $\gamma : I \to M$ is a geodesic if it is locally distance minimizing.

(v) Let $P, Q \in \mathbb{H}$.

(a) If $Re\, P = Re\, Q$, then there is a unique geodesic from P to Q given by the vertical line segment from P to Q.

(b) If $Re\, P \neq Re\, Q$, then there is a unique geodesic from P and Q given by the arc joining P to Q of the unique line in $\mathbb{H}$ (semicircle) with center in $\bar{\mathbb{R}}$ and passing through P and Q.

(vi) All vertical lines are geodesics in $\mathbb{H}$. Moreover, if $b > a$, then for points $x + ia, x + ib \in \mathbb{H}$, we have

$$d_{\mathbb{H}}(x + ia, x + ib) = \log(b/a).$$

(vii) Let $\mathrm{Isom}^+(\mathbb{H})$ denote the group of orientation-preserving isometries of $\mathbb{H}$. Then

$$\mathrm{Isom}^+(\mathbb{H}) \cong \mathrm{PSL}(2, \mathbb{R}).$$

(viii) Given points $P, Q \in \mathbb{H}$, let P' and Q' be the end points in $\bar{\mathbb{R}}$ of the unique geodesic in $\mathbb{H}$ joining P to Q. Then

$$d_{\mathbb{H}}(P, Q) = \log[P', Q, P, Q'].$$

(ix) Given two points $z_1, z_2, \in \mathbb{H}$, we have

(a) $d_{\mathbb{H}}(z_1, z_2) = \log \dfrac{|z_1 - \bar{z_2}| + |z_1 - z_2|}{|z_1 - \bar{z_2}| - |z_1 - z_2|}$.

(b) $\cosh d_{\mathbb{H}}(z_1, z_2) = 1 + \dfrac{|z_1 - z_2|^2}{2\mathrm{Im}z_1 \mathrm{Im}z_2}$.

3.2 The Poincaré disk model $\mathbb{D}$

(i) As a set, the *Poincaré disk* $\mathbb{D}$ is defined by

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

(ii) The metric in the *Poincaré disk model* is defined by

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2}.$$

(iii) If $\gamma : [a, b] \to \mathbb{D}$ is a path in $\mathbb{D}$ that is parametrized in $[a, b]$ with $\gamma(t) = x(t) + iy(t)$, then the *length* $\ell(\gamma)$ *of the path* γ is defined by

$$\ell_{\mathbb{D}}(\gamma) := \int_a^b \frac{2}{1 - r^2} \sqrt{\left(\frac{dr}{dt}\right)^2 + r\left(\frac{d\theta}{dt}\right)^2} dt.$$

(iv) Given two points $P, Q \in \mathbb{H}$, the *distance* $d_{\mathbb{D}}(P,Q)$ between P and Q is defined by

$$d_{\mathbb{D}}(P,Q) := \inf \ell_{\mathbb{D}}(\gamma),$$

where the infimum is taken over all paths joining P and Q.

(v) Let $\text{Möb}^+(\mathbb{D}) = \{m \in \text{Möb}^+(\hat{\mathbb{C}}) : m(\mathbb{D}) = \mathbb{D}\}$. Each $m \in \text{Möb}^+(\mathbb{D})$ has the form

$$m(z) = \frac{e^{i\theta}(z-a)}{1-\bar{a}z} \text{ where } a \in \mathbb{D},$$

or equivalently has the form

$$m(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \text{ with } |a|^2 - |b|^2 = 1.$$

Consequently,

$$\text{Möb}^+(\mathbb{D}) \cong \text{PSU}(1,1).$$

(vi) The *Cayley transformation* $C : \mathbb{H} \to \mathbb{D}$ defined by

$$z \overset{C}{\mapsto} \frac{z-i}{z+i}$$

is a conformal isometry.

(vii) Let $P, Q \in \mathbb{D}$.

(a) If P, Q are on the same diameter of $\mathbb{D}$, then the unique geodesic in $\mathbb{D}$ joining P to Q is given by the Euclidean line segment joining P to Q (along the diameter).

(b) If P, Q do not lie on the same diameter, then the unique geodesic in $\mathbb{D}$ joining P to Q is the arc of the circle orthogonal to $S^1 = \partial\,\mathbb{D}$ joining P to Q.

(viii) All radial lines are geodesics in $\mathbb{D}$. In particular, given $a \in \mathbb{D}$, we have

$$d_{\mathbb{D}}(0,a) = \log\left(\frac{1+|a|}{1-|a|}\right).$$

(ix) Given points $P, Q \in \mathbb{D}$, let P' and Q' be the end points in S^1 of the unique geodesic in $\mathbb{D}$ joining P to Q. Then

$$d_{\mathbb{D}}(P,Q) = \log[P', Q, P, Q'].$$

(x) Given two points $z_1, z_2 \in \mathbb{D}$, we have

(a) $d_{\mathbb{D}}(z_1, z_2) = \log \dfrac{|1 - z_1\bar{z}_2| + |z_1 - z_2|}{|1 - z_1\bar{z}_2| - |z_1 - z_2|}$.

(b) $\cosh^2(d_{\mathbb{D}}(z_1, z_2)/2) = \dfrac{|1 - z_1\bar{z}_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}$.

3.3 Properties of hyperbolic space

(i) The spaces $(\mathbb{H}, d_{\mathbb{H}})$ and $(\mathbb{D}, d_{\mathbb{D}})$ have constant negative curvature -1.

(ii) (a) Given $x \in \partial\,\mathbb{H}$, we have $d_{\mathbb{H}}(x, x + ti) = \infty$, for any $t > 0$.

(b) Given $x \in \partial\,\mathbb{D}$, we have $d_{\mathbb{D}}(x, y) = \infty$, for any $y \in \mathbb{D}$.

(iii) The group $\mathrm{Isom}^+(\mathbb{H})$ (or $\mathrm{Isom}^+(\mathbb{D})$) acts transitively on:

(a) $\mathbb{H}$ (or $\mathbb{D}$).

(b) Hyperbolic lines in $\mathbb{H}$ (or $\mathbb{D}$).

(c) Equidistant pairs of points in $\mathbb{H}$ (or $\mathbb{D}$).

(d) Ordered triples in $\partial\,\mathbb{H} = \bar{\mathbb{R}}$ (or $\partial\,\mathbb{D} = S^1$).

(iv) Let $m \in \mathrm{M\ddot{o}b}^+(\mathbb{H})$ be nontrivial. Then it follows from classification of isometries in $\mathrm{M\ddot{o}b}^+(\hat{\mathbb{C}})$ that:

(a) m is parabolic if, and only if m has one fixed point in $\bar{\mathbb{R}}$. Furthermore, m is conjugate in $\mathrm{M\ddot{o}b}(\mathbb{H})$ to the map $q(z) = z + 1$. Equivalently, m is parabolic if, and only if $\mathrm{Trace}^2(m) = 4$.

(b) m is elliptic if, and only if m has one fixed point in $\mathbb{H}$. Furthermore, m is conjugate in $\mathrm{M\ddot{o}b}^+(\mathbb{H})$ to a rotation by θ (i.e a map of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, for some $\theta \in \mathbb{R}$). Equivalently, m is elliptic if, and only if $\mathrm{Trace}^2(m) < 4$.

(c) m is loxodromic if, and only if m has two fixed points in $\bar{\mathbb{R}}$. Furthermore, m is conjugate in $\mathrm{M\ddot{o}b}^+(\mathbb{H})$ to the map $q(z) = kz$, for some $k > 0$. Equivalently, m is hyperbolic if, and only if $\mathrm{Trace}^2(m) > 4$.

(v) Let $C(z_0, r)$ denoted the hyperbolic circle with center $z_0 \in \mathbb{D}$ and radius $r > 0$. Then $C(0, r)$ coincides with a Euclidean circle with center 0 and radius $\rho = \tanh(r/2)$.

(vi) The circumference of a hyperbolic circle in $\mathbb{D}$ of radius ρ is $2\pi \sinh(\rho)$. The area of the hyperbolic disk of radius ρ if $4\pi \sinh^2(\rho/2)$. (Note that both circumference and area grow exponentially with the radius.)

(vii) Since hyperbolic isometries map Euclidean circles to Euclidean circles, the hyperbolic circle $C(z_0, r)$ will coincide with a Euclidean circle, whose center does not necessarily coincide with the hyperbolic center. As this reasoning extends to hyperbolic disks enclosed by these circles, the topologies $(\mathbb{D}, d_{\mathbb{D}})$ and $\mathbb{R}^2$ have the same basic open sets, and hence they are homeomorphic.

(viii) There exists a unique perpendicular from a point $P \in \mathbb{D}$ (or $\mathbb{H}$) to a hyperbolic line $L \subset \mathbb{D}$ (or $\mathbb{H}$) that realizes the distance between them.

(ix) A perpendicular projection onto a hyperbolic line L in $\mathbb{H}$ (or $\mathbb{D}$) strictly reduces the distance between points.

(x) Let L and L' be disjoint hyperbolic lines which do not meet at $\partial\,\mathbb{H}$ (or $\partial\,\mathbb{D}$). Then L and L' have a unique common perpendicular that realizes the distance between them. Moreover, if the two lines have a common end point in $\partial\,\mathbb{H}$ (or $\partial\,\mathbb{D}$), then $d_{\mathbb{H}}(L, L') = 0$.

(xi) The set of all points in $\mathbb{H}$ (or $\mathbb{D}$) which are at a fixed distance d from a given line L is a circle in $\hat{\mathbb{C}}$ having the same endpoints as L on $\partial\,\mathbb{H}$ (or $\partial\,\mathbb{D}$) making an angle $\theta = \theta(d)$ that is uniquely determined by d.

(xii) An *horocycle* is the limit of a hyperbolic circle as its center approaches $\partial\,\mathbb{H}$.

(xiii) In $\mathbb{H}$, horocircles are either:

  (a) Horizonal lines, if the tangency point is ∞, or

  (b) Circles that are tangent to $\mathbb{R}$.

(xiv) The length of a horocircle between two points is exponentially larger than the hyperbolic distance between them.

3.4 Hyperbolic trigonometry

(i) The sum of the angles of a hyperbolic triangle is strictly less than π. Consequently, the sum of the angles of a hyperbolic n-gon is strictly less than $(n-2)\pi$.

(ii) Let P be a point that is at a distance d from a hyperbolic line L. Then there is a limiting value θ to the angle made by lines L' through P (with the perpendicular from P to L) not meeting L called the *angle of parallelism*.

(iii) The angle of parallelism θ can be computed by considering a triangle with angles 0, $\pi/2$, θ. In such a triangle, we have

$$\cosh d = \csc \theta.$$

Equivalently, we have

$$\sinh d = \cot \theta \text{ or } \tanh d = \cos \theta.$$

(iv) There is an upper bound to the length of the altitude of any hyperbolic isosceles right-angled triangle called the *Schweikart's constant*, which is given by $\log(1+\sqrt{2})$.

(v) (Pythagoras Theorem) In a right angled hyperbolic triangle whose sides have lengths a, b, and c, where c is the hypotenuse, we have

$$\cosh c = \cosh a \cosh b.$$

(vi) (Gauss-Bonnet Theorem) The area of a hyperbolic triangle with angles α, β, and γ is given by

$$\pi - (\alpha + \beta + \gamma).$$

Consequently, the area a hyperbolic n-gon with internal angles α_i, for $1 \leq i \leq n$ is

$$(n-2)\pi - \sum_{i=1}^{n} \alpha_i.$$

(vii) Two hyperbolic triangles T and T' are *congruent* if there exists $m \in \text{Möb}(\mathbb{H})$ such that $m(T) = T'$.

(viii) Any two hyperbolic triangles with the same internal angles are congruent.

(ix) The following conditions for congruency of triangles in Euclidean geometry also hold true in hyperbolic geometry:

$$AAA,\ SAS,\ SSS,\ SAA,\ \text{and}\ RHS.$$

(x) For any three real numbers α, β, $\gamma \geq 0$ with $\alpha+\beta+\gamma < \pi$, there exists an hyperbolic triangle with these numbers as internal angles.

(xi) For $n \geq 3$ and $\theta \in (0, (n-2)\pi/n)$, there exists a regular hyperbolic n-gon with internal angle θ.

(xii) Let ABC be a hyperbolic triangle with sides of lengths a, b, c opposite to internal angles α, β, γ at vertices A, B, C respectively. Then

(a) If $\gamma = \pi/2$, then

$$\cos\beta = \tanh a/\tanh c,\ \sin\beta = \sinh b/\sinh c,\ \tan\beta = \tanh b/\sinh a.$$

(b) The *hyperbolic sine law* is given by

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}.$$

(c) The *first hyperbolic cosine law* is given by

$$\cos\gamma = \frac{\cosh a\cosh b - \cosh c}{\sinh a\sinh b}.$$

(d) The *second hyperbolic cosine law* is given by

$$\cos c = \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

4 Introduction to hyperbolic surfaces

4.1 Hyperbolic structures on surfaces

(i) A *hyperbolic surface* is a smooth surface with a Riemannian metric such that each point on the surface has a neighborhood that is isometric to an open neighborhood of $\mathbb{H}$.

(ii) A *hyperbolic structure* on a surface S is an atlas of charts on S such that:

  (a) The image of every coordinate chart is homeomorphic to a disk in $\mathbb{H}$.

  (b) The overlap functions are hyperbolic isometries.

  (c) The atlas is maximal.

(iii) A *convex geodesic polygon* is a convex subset of $\mathbb{H}$ whose boundary is a simple closed path of hyperbolic geodesic line segments.

(iv) Let P be a convex geodesic polygon. A labeling for each edge of P by a letter (or a symbol) and an arrow (a direction) is called a *decoration for P*.

(v) A *gluing recipe* for a hyperbolic surface is a finite list $\{P_1, \ldots, P_n\}$ of decorated polygons such that:

  (a) Every symbol (or letter) used as a label appears exactly twice.

  (b) If two edges have the same label, then they have the same hyperbolic length.

  (c) Any complete circuit adds up to 2π.

(vi) Any gluing recipe gives rise to a surface with a hyperbolic structure (i.e. a hyperbolic surface).

(vii) Examples.

(a) For $g \geq 2$, consider a regular convex decorated hyperbolic $4g$-gon P_g with edges labeled using the letters in $\{a_i, b_i \,|\, 1 \leq i \leq g\}$ such that $\partial P_g = \prod_{i=1}^{g} [a_i, b_i]$. This decorated polygon P_g gives rise to a hyperbolic structure on the closed orientable surface S_g of genus g.

(b) For $g \geq 2$, consider a regular convex decorated hyperbolic $4g+2$-gon P_g with edges labeled using the letters in $\{a_i |\, 1 \leq i \leq 2g+1\}$ such that $\partial P_g = \prod_{i=1}^{2g+1} a_i \prod_{i=1}^{2g+1} a_i^{-1}$. This decorated polygon P_g gives rise to another hyperbolic structure on S_g that is non-isometric to the structure in (a).

4.2 Geodesic triangulations and the Gauss-Bonnet Theorem

(i) Let $X \subset \mathbb{H}$ be a finite set. For each $p \in X$, let

$$N_p = \{y \in \mathbb{H} : d_{\mathbb{H}}(y, p) = d_{\mathbb{H}}(x, y), \forall x \in X\}.$$

Then

(a) N_p is convex.

(b) If N_p is bounded, then N_p is the interior of a convex hyperbolic polygon.

(ii) A *geodesic triangulation* of a hyperbolic surface is a decomposition of the surface into a finite union of hyperbolic geodesic triangles.

(iii) Every hyperbolic surface has a geodesic triangulation.

(iv) (Gauss-Bonnet) The hyperbolic area $A(S)$ of a compact hyperbolic surface S is given by

$$A(S) = -2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of the surface. In particular, if S is hyperbolic, then $\chi(S) < 0$.

(v) For $g \geq 0$, let $S_{g,b}$ be the surface of genus g with b boundary components (i.e. with b disjoint disks removed). If $S_{g,b}$ is hyperbolic, then

$$A(S_{g,b}) = -2\pi(2 - 2g - b).$$

Consequently, if $S_{g,b}$ is hyperbolic, then $2g + b > 2$.

4.3 The universal cover of a hyperbolic surface

(i) A *Riemannian cover* of a Riemannian manifold X is a Riemannian manifold $\tilde{X}$ such that the covering map $p : \tilde{X} \to X$ is a local isometry.

(ii) Suppose that X is a Riemannian manifold with a covering space $p : \tilde{X} \to X$. Then there exists a unique (namely the pullback under p) on $\tilde{X}$ such that $p : \tilde{X} \to X$ is a local isometry.

(iii) Let $p : \tilde{X} \to X$ be a Riemannian covering space. If X is complete, then so is $\tilde{X}$.

(iv) (Hadamard). Let X be a complete simply connected surface that is locally isometric to $\mathbb{H}$. Then X is globally isometric to $\mathbb{H}$.

(v) A complete hyperbolic surface is universally covered by $\mathbb{H}$.

(vi) A simple closed curve c on a hyperbolic surface $S = S_{g,b}$ is called *essential*, if its free homotopy class $[c]$ does not contain the trivial curve or any of the components of $\partial S_{g,b}$ as its representatives.

(vii) Every free homotopy class $[c]$ of an essential simple closed c on a hyperbolic surface $S = S_{g,b}$ has a unique geodesic representative.